Tutorial 4

Techniques of Differentiation

Outline

- The Product and Quotient Rules
- The Chain Rule and the General Power Rule
- Implicit Differentiation

The Product & Quotient Rules

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

The Product Rule

EXAMPLE

Differentiate the function.

$$\left(x^2+3\right)\left(x^2-3\right)^{10}$$

SOLUTION

Let $f(x)=x^2+3$ and $g(x)=\left(x^2-3\right)^{10}$. Then, using the product rule, and the general power rule to compute $g'(x)$,

$$\frac{d}{dx}\left[\left(x^2+3\right)\left(x^2-3\right)^{10}\right]=\left(x^2+3\right)\cdot\frac{d}{dx}\left[\left(x^2-3\right)^{10}\right]+\left(x^2-3\right)^{10}\cdot\frac{d}{dx}\left(x^2+3\right)$$

$$=\left(x^2+3\right)\cdot 10\left(x^2-3\right)^9\cdot\frac{d}{dx}\left(x^2-3\right)+\left(x^2-3\right)^{10}\cdot 2x$$

$$=\left(x^2+3\right)\cdot 10\left(x^2-3\right)^9\cdot 2x+\left(x^2-3\right)^{10}\cdot 2x.$$

The Quotient Rule

EXAMPLE

Differentiate.

$$\frac{x^4 - 4x^2 + 3}{x}$$

SOLUTION

Let $f(x) = x^4 - 4x^2 + 3$ and $g(x) = x$. Then, using the quotient rule

$$\frac{d}{dx}\left[\frac{x^4 - 4x^2 + 3}{x}\right] = \frac{x \cdot \frac{d}{dx}\left(x^4 - 4x^2 + 3\right) - \left(x^4 - 4x^2 + 3\right) \cdot \frac{d}{dx}(x)}{x^2}$$

$$= \frac{x \cdot \left(4x^3 - 8x\right) - \left(x^4 - 4x^2 + 3\right) \cdot 1}{x^2}$$

Now simplify

$$= \frac{3x^4 - 4x^2 - 3}{x^2} = \frac{3x^4}{x^2} - \frac{4x^2}{x^2} - \frac{3}{x^2} = 3x^2 - 4 - 3x^{-2}$$

The Quotient Rule

CONTINUED

Now let's differentiate again, but first simplify the expression.

$$\frac{x^4 - 4x^2 + 3}{x} = \frac{x^4}{x} - \frac{4x^2}{x} + \frac{3}{x}$$

$$= x^3 - 4x + 3x^{-1}$$

Now we can differentiate the function in its new form.

$$\frac{d}{dx}\left(x^3 - 4x + 3x^{-1}\right) = 3x^2 - 4 - 3x^{-2}$$

Notice that the same answer was acquired both ways.

The Product Rule & Quotient Rule

Another way to order terms in the product and quotient rules, for the purpose of memorizing them more easily, is

PRODUCT RULE

$$\frac{d}{dx}\left[f(x)g(x)\right] = f'(x)g(x) + g'(x)f(x)$$

QUOTIENT RULE

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - g'(x)f(x)}{\left[g(x)\right]^2}.$$

The Chain Rule

The Chain Rule To differentiate $f(g(x))$, first differentiate the outside function $f(x)$ and substitute $g(x)$ for x in the result. Then multiply by the derivative of the inside function $g(x)$. Symbolically,

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).$$

e.g. $$\frac{d}{dx}\left(x^3+4x+5\right)^{11} = 11\left(x^3+4x+5\right)^{10} \cdot \left(3x^2+4\right)$$

The Chain Rule

EXAMPLE

Use the chain rule to compute the derivative of $f(g(x))$, where $f(x)=\frac{4}{x}+x^2$ and $g(x)=1-x^4$.

SOLUTION

$$f'(x)=-\frac{4}{x^2}+2x, \qquad g'(x)=-4x^3$$

$$f'(g(x))=-\frac{4}{(1-x^4)^2}+2(1-x^4)$$

Finally, by the chain rule,

$$\frac{d}{dx}f(g(x))=f'(g(x))g'(x)=\left[-\frac{4}{(1-x^4)^2}+2(1-x^4)\right](-4x^3).$$

The Chain Rule

EXAMPLE

Compute $\frac{dy}{dx}$ using the chain rule.

$$y = \sqrt{u+1},\ u = 2x^2$$

SOLUTION

Since y is not given directly as a function of x, we cannot compute $\frac{dy}{dx}$ by differentiating y directly with respect to x. We can, however, *differentiate with respect to u* the relation $y = \sqrt{u+1}$, and get

$$\frac{dy}{du} = \frac{1}{2\sqrt{u+1}}.$$

Similarly, we can *differentiate with respect to x* the relation $u = 2x^2$ and get

$$\frac{du}{dx} = 4x.$$

The Chain Rule

CONTINUED

Applying the chain rule, we obtain

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{1}{2\sqrt{u+1}} \cdot 4x.$$

It is usually desirable to express $\frac{dy}{dx}$ as a function of x alone, so we substitute $2x^2$ for u to obtain

$$\frac{dy}{dx} = \frac{4x}{2\sqrt{2x^2+1}}.$$

Implicit Differentiation

Finding $\frac{dy}{dx}$ by Implicit Differentiation

Differentiate each term of the equation *with respect to* x, treating y as a function of x.

Move all terms involving $\frac{dy}{dx}$ to the left side of the equation and move the other terms to the right side.

Factor out $\frac{dy}{dx}$ on the left side of the equation.

Divide both sides of the equation by the factor that multiplies $\frac{dy}{dx}$.

Implicit Differentiation

EXAMPLE

Use implicit differentiation to determine the slope of the graph at the given point.

$$4y^3 - x^2 = -5; \;\; x = 3, y = 1$$

SOLUTION

The second term, x^2, has derivative $2x$ as usual. We think of the first term, $4y^3$, as having the form $4[g(x)]^3$. To differentiate we use the chain rule:

$$4\frac{d}{dx}[g(x)]^3 = 12[g(x)]^2 g'(x)$$

or, equivalently,

$$\frac{d}{dx}\left(4y^3\right) = 12y^2 \frac{dy}{dx}.$$

Implicit Differentiation

CONTINUED

On the right side of the original equation, the derivative of the constant function -5 is zero. Thus implicit differentiation of $4y^3 - x^2 = -5$ yields

$$12y^2 \frac{dy}{dx} - 2x = 0.$$

Solving for $\frac{dy}{dx}$ we have

$$\frac{dy}{dx} = \frac{2x}{12y^2} = \frac{x}{6y^2}.$$

At the point (3, 1) the slope is

$$\left.\frac{dy}{dx}\right|_{\substack{x=3\\y=1}} = \left.\frac{x}{6y^2}\right|_{\substack{x=3\\y=1}} = \frac{(3)}{6(1)^2} = \frac{3}{6} = \frac{1}{2}.$$

Implicit Differentiation

This is the general power rule for implicit differentiation:

$$\frac{d}{dx}y^r = ry^{r-1}\frac{dy}{dx}$$

EXAMPLE

Use implicit differentiation to determine $\frac{dy}{dx}$ for $x^2+4xy+4y=1$

SOLUTION

$$x^2+4xy+4y=1$$ This is the given equation.

$$\frac{d}{dx}\left(x^2+4xy+4y\right)=\frac{d}{dx}(1)$$ Differentiate.

$$\frac{d}{dx}\left(x^2\right)+\frac{d}{dx}(4xy)+\frac{d}{dx}(4y)=\frac{d}{dx}(1)$$ Eliminate the parentheses.

$$2x+\frac{d}{dx}(4xy)+4\frac{dy}{dx}=0$$ Differentiate all but the second term.

Implicit Differentiation

CONTINUED

$$2x+\left[4x\frac{d}{dx}(y)+y\frac{d}{dx}(4x)\right]+4\frac{dy}{dx}=0$$

Use the product rule on the second term where $f(x) = 4x$ and $g(x) = y$.

$$2x+4x\frac{dy}{dx}+4y+4\frac{dy}{dx}=0$$

Differentiate.

$$4x\frac{dy}{dx}+4\frac{dy}{dx}=-2x-4y$$

Subtract so that the terms *not* containing dy/dx are on one side.

$$(4x+4)\frac{dy}{dx}=-2x-4y$$

Factor.

$$\frac{dy}{dx}=\frac{-2x-4y}{4x+4}$$

Divide.

Implicit Differentiation

EXAMPLE

Find the slope of the tangent line to the graph of $x^2 + 3xy - 5y = 5$ at the point (2, 1)

SOLUTION

$$x^2 + 3xy - 5y = 5$$ This is the given equation.

$$\frac{d}{dx}\left(x^2 + 3xy - 5y\right) = \frac{d}{dx}(5)$$ Differentiate.

$$2x + 3xy' + 3y - 5y' = 0$$ Eliminate the parentheses.

$$4 + 6y' + 3 - 5y' = 0$$ Substitute.

$$y' = -7$$